Stationary Strings and Principal Killing Triads in 2+1 Gravity

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Abstract

A new tool for the investigation of $2+1$ dimensional gravity is proposed. It is shown that in a stationary $2+1$ dimensional spacetime, the eigenvectors of the covariant derivative of the timelike Killing vector form a rigid structure, the *principal Killing triad*. Two of the triad vectors are null, and in many respects they play the role similar to the principal null directions in the algebraically special 4-D spacetimes. It is demonstrated that the principal Killing triad can be efficiently used for classification and study of stationary $2+1$ spacetimes.

One of the most interesting applications is a study of minimal surfaces in a stationary spacetime. A *principal Killing surface* is defined as a surface formed by Killing trajectories passing through a null ray, which is tangent to one of the null vectors of the principal Killing triad. We prove that a principal Killing surface is minimal if and only if the corresponding null vector is geodesic. Furthermore, we prove that if the $2+1$ dimensional spacetime contains a static limit, then

the only regular stationary timelike minimal 2-surfaces that cross the static limit, are the minimal principal Killing surfaces.

A timelike minimal surface is a solution to the Nambu-Goto equations of motion and hence it describes a cosmic string configuration. A stationary string interacting with a $2+1$ dimensional rotating black hole is discussed in detail.

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1 Introduction

Gravity in $2+1$ dimensions has been intensively investigated in recent years, largely because it provides a relatively simple laboratory for testing ideas developed for the much more complicated problems of realistic $3+1$ dimensional gravity; for a recent review see [1].

It is well-known that the Weyl tensor vanishes in $2+1$ dimensions, so that the Riemann tensor can be written explicitly in terms of the Ricci tensor. This means that vacuum solutions to the Einstein equations are locally flat; any deviation from flat $2+1$ dimensional Minkowski spacetime must be merely topological. This is no longer true, of course, when matter sources are admitted. Indeed, there is now a long list of known $2+1$ dimensional spacetimes with non-trivial geometrical and topological structures; for reviews see Refs.[2, 3]. Furthermore, many of these $2+1$ dimensional spacetimes (e.g. some black hole spacetimes) are believed to be able to provide important clues towards a better understanding of "related" $3+1$ dimensional spacetimes.

In $2+1$ dimensions there is actually a three-tensor which in some respects plays the same role as the Weyl tensor in higher dimensions; the so-called Cotton tensor [4] (for some early applications, see for instance [5, 6]). A classification of $2+1$ dimensional spacetimes, according to the eigenvalues of the Cotton tensor, was carried out in Ref.[2]. In this paper we propose another classification which works for stationary $2+1$ spacetimes and which can be used instead of the Petrov classification.

One of the aims of the present paper is to demonstrate that in a stationary $2+1$ dimensional spacetime, there is a uniquely defined rigid structure, the *principal Killing triad*: provided that the covariant derivative of the timelike Killing vector has a non-zero eigenvalue, the three independent eigenvectors

constitute a basis for the spacetime. Up to normalization, this basis is defined uniquely. The triad consists of two linearly independent null vectors $l_{\pm}$, and a unit spacelike vector m orthogonal to them.

We first express the geometrical structures (metric, Killing vector, twist,...) in terms of the principal Killing triad. We then obtain some general conditions for the class of $2+1$ dimensional stationary spacetimes for which one or both of the two principal Killing null eigenvectors are geodesics. It is interesting to notice that most of the $2+1$ dimensional spacetimes of physical interest studied in the literature actually fall in this class (Section 2).

A further goal of this paper is the study of 2 dimensional minimal surfaces embedded in the $2+1$ dimensional spacetimes. Such surfaces are interesting because they are solutions of the Nambu-Goto equations of motion and hence describe the interaction of a cosmic string with a background gravitational field, in the limit where the cosmic string is infinitely thin. For this purpose we define a *principal Killing surface* as a stationary 2-surface formed by Killing trajectories passing through a null ray tangent to one of the null vectors of the principal Killing triad. We prove that a principal Killing surface is minimal if and only if the corresponding null vector is geodesic (Section 3). Furthermore, we prove a uniqueness theorem, that is, if the $2+1$ dimensional stationary spacetime contains a static limit, then the only regular stationary timelike minimal 2-surfaces that cross the static limit, are the minimal principal Killing surfaces. We also discuss possible applications of this result (Section 4).

Finally, as a special example, we consider the $2+1$ dimensional black hole anti de Sitter spacetime [7], using our general formalism. We find all regular stationary minimal timelike 2-surfaces (string world-sheets) in this background. Particularly interesting are the strings which cross the static

limit. It turns out that the induced geometry on the world-sheets of these configurations corresponds to global $1+1$ anti de Sitter spacetime. We discuss the physical implications of this result, and we consider the equation determining the propagation of perturbations (*stringons*) along these strings (Section 5).

2 The Principal Killing Triad

In this section we introduce the notion of the principal Killing triad for a stationary $2+1$ dimensional manifold. We choose spacetime coordinates $x^\mu = (t, x^i);\ \mu = 0,1,2;\ i = 1,2$, so that the Killing vector ξ^μ is given by $\xi^\mu = \delta^\mu_t$. A $2+1$ dimensional stationary metric has the form:

$$g_{\mu\nu} = \begin{pmatrix} -F & -FA_i \\ -FA_i & \frac{H_{ij}}{F} - FA_iA_j \end{pmatrix}, \tag{2.1}$$

where $\partial_t F = 0,\ \ \partial_t A_i = 0,\ \ \partial_t H_{ij} = 0$ and $\xi^\mu \xi_\mu = -F$. We also assume that ξ^μ is timelike ($F > 0$) - at least in some open region of the manifold under consideration.

The Killing equation,

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0, \tag{2.2}$$

implies that the tensor $\xi_{\mu;\nu}$ is anti-symmetric. The explicit form of $\xi_{\mu;\nu}$ in the metrics (2.1) can be easily obtained in general, see Appendix A, but it is not important here. Since $\xi_{\mu;\nu}$ is an anti-symmetric 3×3 matrix, it has eigenvalues $(-\kappa, +\kappa, 0)$. Denote the corresponding eigenvectors as $(l^\mu_+, l^\mu_-, m^\mu)$:

$$\xi_{\mu;\nu} l^\nu_\pm = \mp\kappa l_{\pm\mu}, \qquad \xi_{\mu;\nu} m^\nu = 0. \tag{2.3}$$

We would like to stress that a set of three eigenvectors fulfilling (2.3) can be obtained for any (not only timelike) Killing vector of a $2+1$ dimensional

metric. However, in the rest of this paper we consider the Killing vector $\xi^\mu = \delta^\mu_t$ and the metric (2.1).

In the general case κ is either real or pure imaginary. We restrict ourselves by considering the case when $\kappa = \kappa(x^i)$ is real and non-zero - at least in some open region of the manifold under consideration. This turns out to be the case for most $2+1$ dimensional stationary metrics of physical interest, and we shall return to the physical explanation of this, in a moment. Taking then $\kappa^2 > 0$, it follows that $(l^\mu_+, l^\mu_-, m^\mu)$ are linearly independent vectors, and thus provide a basis for the manifold. We will call this basis the "principal Killing triad". From equations (2.3), it follows that l_+ and l_- are null vectors, while m is a spacelike vector orthogonal to both of them:

$$g_{\mu\nu} l^\mu_\pm l^\nu_\pm = 0, \qquad g_{\mu\nu} l^\mu_\pm m^\nu = 0. \tag{2.4}$$

Without loss of generality, we normalize the principal Killing triad so that:

$$g_{\mu\nu} l^\mu_\pm \xi^\nu = -1, \qquad g_{\mu\nu} m^\mu m^\nu = 1. \tag{2.5}$$

With this normalization, the principal Killing triad is uniquely defined for a generic stationary 2+1 dimensional metric, assuming only that the eigenvalue κ is real and non-zero.

The Killing vector ξ takes the following form when expressed in the basis of the principal Killing triad:

$$\xi^\mu = \frac{-1}{l_+ \cdot l_-}(l^\mu_+ + l^\mu_-) + (m \cdot \xi) m^\mu. \tag{2.6}$$

By covariant differentiation of the identity $\xi^\mu \xi_\mu = -F$, and comparison with equations (2.3)-(2.4), we get:

$$\kappa = -\frac{1}{2}\frac{dF}{dx^\nu} l^\nu_+ = \frac{1}{2}\frac{dF}{dx^\nu} l^\nu_- \tag{2.7}$$

The anti-symmetric tensor $\xi_{\mu;\nu}$ written in terms of the principal Killing triad reads:

$$\xi_{\mu;\nu} = -\frac{\kappa}{l_+ \cdot l_-}(l_{+\mu}l_{-\nu} - l_{+\nu}l_{-\mu}), \tag{2.8}$$

while the metric (2.1) is given by:

$$g_{\mu\nu} = m_\mu m_\nu + \frac{l_{+\mu}l_{-\nu} + l_{+\nu}l_{-\mu}}{l_+ \cdot l_-}. \tag{2.9}$$

The normalization (2.5) results in the following relationship:

$$-F = (m \cdot \xi)^2 + \frac{2}{l_+ \cdot l_-}. \tag{2.10}$$

If a metric is stationary, but not static, the twist Ω, defined by,

$$\Omega \equiv -\frac{1}{2\xi^2}e^{\mu\nu\rho}\xi_{\mu;\nu}\xi_\rho, \tag{2.11}$$

does not vanish (here $e^{\mu\nu\rho}$ is the anti-symmetric tensor in 3 dimensions). The twist Ω is the angular velocity of the rotation of the local Killing frame. Using equations (2.6), (2.8) we can write the twist Ω as:

$$\Omega = \frac{\kappa}{F}\, m \cdot \xi. \tag{2.12}$$

Thus the scalar product $m \cdot \xi$ is a measure of the "rotation" of the stationary spacetime: a stationary spacetime with non-zero κ is static if and only if $m \cdot \xi = 0$. The twist Ω can also be presented as:

$$\Omega^2 = \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}, \tag{2.13}$$

where $\omega_{\mu\nu}$ is the velocity of rotation of the Killing observer:

$$\omega_{\mu\nu} = \frac{1}{\sqrt{-\xi^2}}\left(\xi_{\mu;\nu} - (\xi_\mu a_\nu - \xi_\nu a_\mu)\right), \tag{2.14}$$

and a_μ is the acceleration of the Killing observer:

$$a_\mu = \frac{1}{2}\nabla_\mu \log(-\xi^2) = \frac{-1}{F}\xi^\nu \xi_{\nu;\mu}. \tag{2.15}$$

Notice that:

$$\kappa^2 = -\frac{1}{2}\xi_{\mu;\nu}\xi^{\mu;\nu}, \tag{2.16}$$

and therefore:

$$\kappa^2 = F(a_\mu a^\mu - \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}) = F(a_\mu a^\mu - \Omega^2). \tag{2.17}$$

Thus κ^2 is the acceleration squared minus the angular velocity squared. If therefore we restrict ourselves to consider only κ real and non-zero (see the comments after equation (2.3)), we only consider (regions of) spacetimes where the acceleration is larger than the angular velocity (in suitable units).

Let us now examine the null vectors of the principal Killing triad in more detail. For a specific metric, they may or may not be geodesics. In fact, we can classify the stationary $2+1$ dimensional metrics according to whether l_+ and/or l_- are geodesics or not. Since the principal Killing triad provides a basis on the manifold, we have the general expansion:

$$l_\pm^\nu l_{\pm\mu;\nu} = A_\pm l_{\pm\mu} + B_\pm l_{\mp\mu} + C_\pm m_\mu. \tag{2.18}$$

By contracting with $l_\pm^\mu$ we find that $B_\pm = 0$, and by contracting with ξ^μ that $A_\pm = (m\cdot\xi)C_\pm$, that is to say:

$$l_\pm^\nu l_{\pm\mu;\nu} = [(m\cdot\xi)l_{\pm\mu} + m_\mu]C_\pm, \tag{2.19}$$

where $C_\pm$ are given implicitly by:

$$C_\pm = m^\mu l_\pm^\nu l_{\pm\mu;\nu}. \tag{2.20}$$

Explicit expressions for $C_\pm$ can be obtained from equation (2.3):

$$C_\pm = -m^\mu l_\pm^\nu(\frac{1}{\kappa}\xi_{\mu;\rho}l_\pm^\rho)_{;\nu} = -\frac{1}{\kappa}m^\mu l_\pm^\nu l_\pm^\rho \xi_{\mu;\rho;\nu} = -\frac{1}{\kappa}R_{\rho\sigma\mu\nu}m^\mu l_\pm^\nu l_\pm^\rho \xi^\sigma, \tag{2.21}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the manifold. This expression can be rewritten further using the expansion of the Riemann tensor in terms of the Ricci tensor, valid generally in 3 dimensions:

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{R}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \quad (2.22)$$

A congruence of null rays generated by $l_\pm$ is geodesic if and only if $C_\pm$ vanishes, that is if and only if :

$$(m \cdot \xi)R_{\mu\nu}l_\pm^\mu l_\pm^\nu + R_{\mu\nu}l_\pm^\mu m^\nu = 0. \quad (2.23)$$

If we consider a "physical" spacetime obtained as a solution of the Einstein equations,

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = T_{\mu\nu}, \quad (2.24)$$

with some physically relevant energy-momentum tensor $T_{\mu\nu}$, the condition (2.23) is written:

$$T_{\mu\nu}[(m \cdot \xi)l_\pm^\mu l_\pm^\nu + l_\pm^\mu m^\nu] = 0. \quad (2.25)$$

It is also possible to rewrite equation (2.23) as a second order non-linear partial differential equation for the functions (F, A_i, H_{ij}), appearing in the metric (2.1), but we shall not consider this equation and its general solution here. For our purposes, it will be sufficient to consider some families of special solutions to equation (2.23). It happens that these families of solutions cover most of the known $2+1$ dimensional stationary spacetimes of physical interest.

A family of solutions to Einsteins equations which trivially fulfills equation (2.23) is obtained from an energy-momentum tensor of the form:

$$T_{\mu\nu} = f(x^i)g_{\mu\nu}, \quad (2.26)$$

where $f(x^i)$ is an arbitrary function. Particular examples in this family of $2+1$ dimensional solutions are the de Sitter and anti de Sitter spacetimes [8].

More generally, consider the following family of metrics:

$$g_{\mu\nu} = \begin{pmatrix} -F(x) & 0 & -F(x)A_y(x) \\ 0 & \frac{H_{xx}(x)}{F(x)} & 0 \\ -F(x)A_y(x) & 0 & \frac{H_{yy}(x)}{F(x)} - F(x)A_y^2(x) \end{pmatrix}; \qquad x^\mu = \begin{pmatrix} t \\ x \\ y \end{pmatrix}. \tag{2.27}$$

These are a special case of the metric (2.1) where an additional spacelike Killing vector ∂_y is present. By explicit construction of the principal Killing triad in this family of metrics, it can be shown that equation (2.23) is automatically fulfilled for *arbitrary* functions $F(x), H_{xx}(x), H_{yy}(x)$ if and only if the function $A_y(x)$ is of the form:

$$A_y(x) = \frac{c_1}{F(x)} + c_2, \tag{2.28}$$

where c_1, c_2 are constants.

We now list some known metrics which belong to this class. Consider first the case when $c_1 = c_2 = 0$. We already mentioned de Sitter and anti de Sitter spacetimes in connection with equation (2.26). When written in static coordinates, they are in this class. This is also the case for the metric around a static massive point particle [9], as well as for the metric around a static closed or open string [10]. The static solutions obtained by minimal coupling to a massless scalar field or by coupling to a static magnetic field (or a stiff perfect fluid) [2] are also in this class. So is the charged black string solution of Horne and Horowitz [11]. Now consider the case when $c_2 = 0$ but $c_1 \neq 0$. In this class we find the metric of a massless spinning point particle [12], as well as the metric of the $2+1$ black hole anti de Sitter (BH-ADS) solution of Banados et al [7]. In both these cases c_1 represents the angular momentum.

Notice that the $2+1$ BH-ADS solution is actually also of the form (2.26) since it is locally (and asymptotically) isometric to $2+1$ dimensional anti de Sitter spacetime. Thus in all these spacetimes both null vectors $l_\pm$ of the principal Killing triad are geodesics.

In the cases where one or both of $l_\pm$ are geodesics, the principal Killing basis is an important tool for the study of stationary minimal 2-surfaces embedded in the higher dimensional stationary spacetime. This was demonstrated in a recent publication [13] concerning the $3+1$ dimensional Kerr-Newman black hole spacetime, using the corresponding principal Killing tetrad. In $2+1$ dimensions, the analytical computations are much simpler, and it is possible to consider generic spacetimes. The principal physical motivation for the study of timelike minimal 2-surfaces is that such surfaces are solutions of the Nambu-Goto equations of motion, and hence can represent string configurations in the spacetime under consideration. We thus consider the $2+1$ dimensional spacetimes as convenient toy-models for the investigation of the physics of stationary string world-sheets embedded in higher dimensions. This is the topic of the next section.

3 String World-Sheets

Our aim in this section is to consider stationary regular timelike minimal 2-surfaces, corresponding to string world-sheets, embedded in stationary $2+1$ dimensional spacetimes where at least one of the two null vectors $l_\pm$ of the principal Killing triad is geodesic. For this purpose we begin by considering the general properties of stationary timelike 2-surfaces embedded in a $2+1$ dimensional stationary spacetime.

Let Σ be a 2 dimensional timelike surface embedded in a $2+1$ dimensional stationary spacetime, and let ξ be the corresponding Killing vector which is

timelike - at least in some open region of the spacetime. Σ is said to be *stationary* if the Killing vector field ξ is everywhere tangent to it. For any such surface Σ there exists two linearly independent null vector fields, say l_1 and l_2, tangent to Σ. We assume that the integral curves of each of the null vectors $l_{1,2}$ form a congruence that covers Σ, i.e. each point $p \in \Sigma$ lies on exactly one of these integral curves.

Thus we can construct a stationary timelike surface Σ in the following way: consider a null ray γ with tangent vector field l. For each point $p \in \gamma$, there is precisely one Killing trajectory with tangent vector ξ that passes through it. The set of Killing trajectories passing through γ forms a stationary 2-surface Σ. We define l over Σ by Lie propagation along each Killing trajectory. We call γ a basic ray of Σ. It is easily verified that l remains null when defined in this manner over Σ.

We can use the Killing time parameter u along Killing trajectories and the affine parameter λ along γ as coordinates on Σ. In these coordinates $\zeta^A = (u, \lambda)$ one has $x^\mu_{,u} = \xi^\mu$ and $x^\mu_{,\lambda} = l^\mu$, and the induced metric $G_{AB} = g_{\mu\nu} x^\mu_{,A} x^\nu_{,B}$ $(A, B, ... = 0, 1)$ is of the form:

$$dS^2 = G_{AB} d\zeta^A d\zeta^B = -F du^2 + 2(\xi \cdot l) du d\lambda. \tag{3.1}$$

Now introduce a spacelike vector n^μ normal to the timelike 2-surface Σ:

$$g_{\mu\nu} n^\mu n^\nu = 1, \qquad g_{\mu\nu} x^\mu_{,A} n^\nu = 0. \tag{3.2}$$

For this choice of the triad (ξ, l, n), the following completeness relation is satisfied:

$$g^{\mu\nu} = G^{AB} x^\mu_{,A} x^\nu_{,B} + n^\mu n^\nu. \tag{3.3}$$

The normal vector n spans the vector space normal to the surface at a given point.

The second fundamental form Ω_{AB} for Σ is defined as:

$$\Omega_{AB} = g_{\mu\nu} n^\mu x^\rho_{,A} \nabla_\rho x^\nu_{,B}. \tag{3.4}$$

The condition that a surface Σ is minimal can be written in terms of the trace of the second fundamental form:

$$\Omega_A{}^A \equiv G^{AB}\Omega_{AB} = 0. \tag{3.5}$$

This is equivalent to the Gauss-Weingarten equation, which for the line-element (3.1) takes the explicit form:

$$2(\xi \cdot l) l^\rho \xi^\mu_{;\rho} + F l^\rho l^\mu_{;\rho} + (\xi \cdot l) l^\rho \frac{\partial}{\partial x^\rho}\left(\frac{F}{\xi \cdot l}\right) l^\mu = 0. \tag{3.6}$$

It is easily verifed that equation (3.6) is invariant under reparametrizations of l^μ, i.e. if l^μ satisfies (3.6) then so does $g(x) l^\mu$, for an arbitrary function $g(x)$. Thus without loss of generality we may normalize l^μ so that $l \cdot \xi = -1$. Then equation (3.6) becomes:

$$-2 l^\rho \xi^\mu_{;\rho} + F l^\rho l^\mu_{;\rho} + l^\rho \frac{\partial F}{\partial x^\rho} l^\mu = 0. \tag{3.7}$$

Consider a special type of a stationary 2-surface embedded in a stationary $2+1$ dimensional spacetime with principal Killing triad $(l^\mu_+, l^\mu_-, m^\mu)$. Namely, consider a surface for which the null vector l coincides with one of the two null vectors $l_\pm$. We call such a surface $\Sigma_\pm$ a *principal Killing surface* and $\gamma_\pm$ its *basic ray*. We shall use indices $\pm$ to distinguish between quantities connected with $\Sigma_\pm$.

In what follows we restrict ourselves by considering timelike principal Killing surfaces. It is motivated by the fact that this class of surfaces is connected with string world-sheets. By comparing equations (2.3), (2.7) with equation (3.7), we immediately get the following theorem:

A timelike principal Killing surface is a minimal surface if and only if the corresponding null vector of the principal Killing triad is geodesic. That is, if and only if the $2+1$ dimensional stationary metric fulfills equation (2.23), for the corresponding null vector.

The theorem is valid in the regions where $\xi^2 \neq 0$. In that case, $\Sigma_\pm$ is a stationary solution of the Nambu-Goto equations. However, it should be stressed that such minimal principal Killing surfaces are only very special stationary minimal surfaces. A general stationary string solution (minimal surface) must be obtained by explicitly solving equation (3.7), which is generally a highly non-trivial problem. However, it turns out that the minimal principal Killing surfaces play a particularly important role if the stationary $2+1$ dimensional spacetime, besides fulfilling equation (2.23), also contains a curve where the Killing vector ξ becomes null. This curve is defined by:

$$F(x^i) = 0, \tag{3.8}$$

and corresponds to a static limit (or a horizon) in the $2+1$ dimensional spacetime. This will be discussed in the following section.

4 Uniqueness Theorem

Consider a stationary $2+1$ dimensional metric, which fulfills equation (2.23) for at least one of the two null vectors of the principal Killing triad. We also assume that there exists a curve where the Killing vector ξ becomes null. This curve is called the static limit curve $S_{\rm st}$ and it is defined by equation (3.8). When constructing a stationary timelike 2-surface Σ (not necessarily minimal) in such a spacetime, following the procedure outlined

in the previous section, we always choose the null vector l to be that of two possible null vector fields $l_{1,2}$ on Σ which does not coincide with the Killing vector ξ at $S_{\rm st}$. In that case, the metric (3.1) is regular on the static limit curve.

We prove now that the only stationary timelike minimal 2-surfaces that cross the static limit curve, and are regular in its vicinity, are the minimal principal Killing surfaces.

Consider a stationary timelike 2-surface Σ with the line element (3.1). The condition that the surface is minimal is given by equation (3.7). Since $l^\rho l^\mu_{;\rho}$ is regular at $S_{\rm st}$, this equation on the static limit curve ($F=0$) reduces to:

$$(\xi_{\mu;\rho} - \frac{1}{2}\frac{\partial F}{\partial x^\rho} l_\mu) l^\rho = 0. \tag{4.1}$$

From equations (2.3), (2.7) it follows that $l \propto l_+$ (or $l \propto l_-$) on the static limit curve.

Now suppose there exists a timelike minimal surface Σ different from Σ_+. On the static limit curve, Σ must have $l \propto l_+$. In the vicinity of the static limit curve, l can have only small deviations from l_+. From the conditions $l \cdot l = 0$ and $l \cdot \xi = -1$, we then get the following general form of l in the vicinity of the static limit curve:

$$l^\mu = [1 + B(m \cdot \xi)] l^\mu_+ + B m^\mu + \mathcal{O}(B^2), \tag{4.2}$$

for some function B, where we have only kept terms up to first order in B. We insert this expression into equation (3.7), contract with m_μ, and keep only terms linear in B :

$$-2 m_\mu l^\rho \xi^\mu_{;\rho} = 0, \qquad \text{(to all orders in } B\text{)}, \tag{4.3}$$

$$m_\mu l^\rho \frac{\partial F}{\partial x^\rho} l^\mu = l^\rho_+ \frac{\partial F}{\partial x^\rho} B + \mathcal{O}(B^2), \tag{4.4}$$

$$m_\mu F l^\rho l^\mu_{;\rho} = F l^\rho_+ \frac{\partial B}{\partial x^\rho} + F m^\rho m^\mu l_{+\mu;\rho} B + \mathcal{O}(B^2). \tag{4.5}$$

Thus altogether:

$$l^\rho_+ \frac{\partial B}{\partial x^\rho} + [\frac{1}{F} l^\rho_+ \frac{\partial F}{\partial x^\rho} + m^\rho m^\mu l_{+\mu;\rho}] B + \mathcal{O}(B^2) = 0. \tag{4.6}$$

Notice that the second term in the bracket corresponds to the "expansion" θ of the null rays [14]:

$$m^\rho m^\mu l_{+\mu;\rho} = l^\mu_{+;\mu} \equiv \theta. \tag{4.7}$$

Since we consider the 2-surface Σ to be timelike and regular, even near the static limit curve, the expansion θ is regular: θ being divergent at some point p, would imply the existence of a caustic. However, this contradicts the fact that only one integral curve of the null-vector l_+ passes through p, as follows from the regularity condition, see [15]. Near the static limit curve the first term in the bracket of equation (4.6) dominates, and it follows that the solution near the static limit curve is approximated by:

$$B = \frac{c}{F}; \qquad c = \text{const.} \tag{4.8}$$

A solution regular near the static limit curve ($F \to 0$) can therefore only be obtained for $c = 0$, which implies that $B = 0$. Thus we have shown that Σ is regular and minimal if and only if $l \propto l_\pm$. This proves the uniqueness theorem: The only stationary timelike minimal 2-surfaces that cross the static limit curve $S_{\rm st}$, and are regular in its vicinity, are the mimimal principal Killing surfaces.

We now discuss possible physical applications of this result.

Consider one of the null vectors of the principal Killing triad, say l_+. Since l^μ_+ cannot be proportional to ξ^μ, either l^x_+ and/or l^y_+ is non-zero (we here use the notation $\mu = (t, x, y)$). Let us assume that (say) $l^x_+ \neq 0$. We also restrict ourselves to consider only metrics where an additional Killing vector (say)

∂_y is present (that is, metrics of the form (2.27)). We can then introduce generalized Eddington-Finkelstein coordinates in the following way:

$$dx_+^\mu = (A_+)^\mu{}_\nu dx^\nu, \tag{4.9}$$

where the matrix $(A_+)^\mu{}_\nu$ is defined by:

$$(A_+)^\mu{}_\nu \equiv \begin{pmatrix} 1 & -l_+^t/l_+^x & 0 \\ 0 & -1/l_+^x & 0 \\ 0 & l_+^y/l_+^x & -1 \end{pmatrix}, \tag{4.10}$$

that is:

$$dt_+ = dt - \frac{l_+^t}{l_+^x}dx, \tag{4.11}$$

$$dx_+ = -\frac{dx}{l_+^x}, \tag{4.12}$$

$$dy_+ = -dy + \frac{l_+^y}{l_+^x}dx. \tag{4.13}$$

In a similar way we can introduce generalized Eddington-Finkelstein coordinates corresponding to l_-.

We have shown that any stationary timelike minimal 2-surface that crosses the static limit curve in the stationary $2+1$ dimensional spacetime (assumed to fulfill equation (2.23)) and remains regular in its vicinity, is a minimal principal Killing surface. For the principal Killing surface Σ_+ we have $\partial x^\mu/\partial\lambda = l_+^\mu$ (up to a constant factor). We can choose the affine parameter λ such that:

$$dx^\mu = -l_+^\mu d\lambda + \xi^\mu du, \tag{4.14}$$

that is:

$$dt = -l_+^t d\lambda + du, \tag{4.15}$$

$$dx = -l_+^x d\lambda, \tag{4.16}$$

$$dy = -l_+^y d\lambda. \tag{4.17}$$

By comparing Equations (4.11)-(4.13) with equations (4.15)-(4.17), we find:

$$dx_+ = d\lambda, \tag{4.18}$$

$$dy_+ = 0, \tag{4.19}$$

$$dt_+ = du. \tag{4.20}$$

Thus a minimal principal Killing surface corresponds to $y_+ =$ const. Physically it means that a stationary string can only cross the static limit curve in very special ways, namely along the direction $y_+ =$ const. The above consideration certainly is also valid for Σ_-.

We can then write down the induced line-element on the world-sheets of $\Sigma_\pm$ in terms of the generalized Eddington-Finkelstein coordinates using equations (2.9), (2.10), (4.14) and (4.18):

$$dS^2 = -F dt_\pm^2 \pm 2 dt_\pm dx_\pm. \tag{4.21}$$

It is an interesting observation that the static limit curve of the $2+1$ dimensional spacetime corresponds to a horizon on the string world-sheet, as follows from equations (3.8), (4.21), *even* if it does not correspond to a horizon in the $2+1$ dimensional spacetime [16]. This may lead to very interesting causal properties [13], as will be discussed in more detail in the next section.

In the general case a minimal surface obtained by a small perturbation of a minimal principal Killing surface is not stationary. A general transverse perturbation about a background Nambu-Goto string world-sheet embedded in $2+1$ dimensions can be written as :

$$\delta x^\mu = \Phi n^\mu, \tag{4.22}$$

where the normal vector n^μ is defined by equations (3.2). The equation defining the perturbation Φ follows from the following effective action [17] (see also [18, 19]):

$$\mathcal{S}_{\text{eff.}} = \int d^2\zeta\sqrt{-G}\;\Phi\left\{G^{AB}\nabla_A\nabla_B + \mathcal{V}\right\}\Phi, \tag{4.23}$$

where $\mathcal{V}$ is a scalar potential defined as:

$$\mathcal{V} \equiv \Omega_{AB}\Omega^{AB} - G^{AB}x^\mu_{,A}x^\nu_{,B}R_{\mu\rho\sigma\nu}n^\rho n^\sigma. \tag{4.24}$$

The equation describing the propagation of perturbations on the world-sheet background is then found to be:

$$\{\Box + \mathcal{V}\}\,\Phi = 0. \tag{4.25}$$

Consider the scalar potential $\mathcal{V} \equiv \Omega_{AB}\Omega^{AB} - G^{AB}x^\mu_{,A}x^\nu_{,B}R_{\mu\rho\sigma\nu}n^\rho n^\sigma$. It is easily verified that the first term vanishes for the minimal principal Killing surfaces $\Sigma_\pm$. The second term is rewritten using equations (2.22), (3.3):

$$\begin{aligned}\mathcal{V} &= R_{\mu\nu}n^\mu n^\nu = R - G^{AB}R_{\mu\nu}x^\mu_{,A}x^\nu_{,B}\\ &= R + FR_{\mu\nu}l^\mu_\pm l^\nu_\pm \pm 2R_{\mu\nu}\xi^\mu l^\nu_\pm.\end{aligned} \tag{4.26}$$

The equation (4.25) of propagation of string perturbations (stringons) can be used to study the stability of the string configurations. One can also expect that if a stationary cosmic string crosses the static limit curve, the Hawking mechanism will provide thermal excitation of quantum stringons.

In Section 5 we shall consider in more detail the equation (4.25) in the case of the $2+1$ dimensional black hole anti de Sitter spacetime.

5 The 2+1 BH-ADS Spacetime

As an important example, we now consider the $2+1$ dimensional BH-ADS spacetime in more detail, using the formalism and tools developed in the

previous sections. This spacetime is often used as a 2+1 dimensional toy model of a rotating black hole. The metric of the $2+1$ dimensional BH-ADS spacetime is given by [7]:

$$ds^2 = (\frac{J^2}{4r^2} - \Delta)dt^2 + \frac{dr^2}{\Delta} - Jdtdr + r^2 d\phi^2, \tag{5.1}$$

where:

$$\Delta = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}. \tag{5.2}$$

Here M represents the mass, J is the angular momentum and the cosmological constant is $\Lambda = -1/l^2$. The Killing vector $\xi^\mu = \delta^\mu_t$, which is timelike at infinity, defines the function F :

$$F = -\xi^2 = \frac{r^2}{l^2} - M. \tag{5.3}$$

For $M^2 l^2 \geq J^2$, there are two horizons ($g_{rr} = \infty$):

$$r_\pm^2 = \frac{Ml^2}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}}\right) \tag{5.4}$$

and a static limit curve ($g_{tt} = 0$):

$$r_{\mathrm{st}}^2 = Ml^2. \tag{5.5}$$

The static limit curve lies outside the external horizon, $r_{\mathrm{st}} \geq r_+$, so the causal structure is similar to that of the 4-dimensional Kerr-Newman geometry. Notice that in $2+1$ dimensions there is no strong curvature singularity at $r = 0$, in fact:

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu}. \tag{5.6}$$

For more details on the local and global geometry of the BH-ADS solution, see for instance [7, 20, 21].

The principal Killing triad, as defined in Section 2, is given by:

$$l^{\mu}_{\pm} = (\frac{1}{\Delta},\ \mp 1,\ \frac{J}{2r^2\Delta}), \qquad m^{\mu} = (0,\ 0,\ \frac{1}{r}). \tag{5.7}$$

The results of Section 2 imply that $l_{\pm}$ are null geodesics:

$$l^{\mu}_{\pm} l_{\pm\mu} = 0, \qquad l^{\mu}_{\pm} l^{\nu}_{\pm;\mu} = 0. \tag{5.8}$$

We are interested in the regular stationary timelike minimal 2-surfaces, corresponding to string world-sheets, embedded in the spacetime (5.1). In the present case we can solve the string equation (3.7) exactly. First we write the null tangent vector l in the general form:

$$l^{\mu} = A l^{\mu}_{+} + B l^{\mu}_{-} + C m^{\mu}, \tag{5.9}$$

for some functions (A, B, C). The two conditions $l \cdot l = 0$ and $l \cdot \xi = -1$ imply:

$$C^2 = \frac{4AB}{\Delta}, \qquad A + B + \frac{J}{2r} C = 1, \tag{5.10}$$

and equation (3.7) reduces to:

$$\frac{dC}{dr} + \left(\frac{1}{r} + \frac{1}{F}\frac{dF}{dr}\right) = 0. \tag{5.11}$$

It follows that:

$$\begin{aligned} A &= \frac{1}{2}\left(1 - \frac{Jk}{2r^2 F}\right) \pm \frac{1}{2}\sqrt{1 - \frac{Jk + k^2}{r^2 F}}, \\ B &= \frac{1}{2}\left(1 - \frac{Jk}{2r^2 F}\right) \mp \frac{1}{2}\sqrt{1 - \frac{Jk + k^2}{r^2 F}}, \\ C &= \frac{k}{rF}, \end{aligned} \tag{5.12}$$

where k is an integration constant. The null tangent vector l is then given by:

$$l^{\mu} = \frac{\partial x^{\mu}}{\partial \lambda} = \left(\frac{1}{\Delta}(1 - \frac{Jk}{2r^2 F}),\ \mp\sqrt{1 - \frac{Jk + k^2}{r^2 F}},\ \frac{1}{r^2\Delta}(k + \frac{J}{2})\right). \tag{5.13}$$

Regular solutions exist in two different situations:

I. If $k^2 + kJ > 0$, then there is a turning point ($dr/d\phi = 0$) outside the static limit:

$$r_0^2 = \frac{1}{2}Ml^2\left(1 + \sqrt{1 + \frac{4(k^2 + kJ)}{M^2l^2}}\right). \tag{5.14}$$

That is to say, the strings are of the "hanging string type" [22], i.e. with both ends at spatial infinity. In the case $J = 0$, these strings were discussed in detail in [23].

II. If $k = 0$, these strings correspond to $l = l_{\pm}$, and according to the uniqueness theorem of Section 4, they are the only strings that cross the static limit curve and being regular in its vicinity. These strings are of the "spiralling string type" [22].

If instead $k^2 + kJ \leq 0$ and $k \neq 0$, the tangent vector l diverges at the static limit curve $F = 0$, that is, the solution is not regular.

We now consider the spiralling strings ($k = 0$) in more detail. In the generalized Eddington-Finkelstein coordinates [24]:

$$dt_{\pm} = dt \pm \frac{dr}{\Delta}, \tag{5.15}$$

$$d\phi_{\pm} = d\phi \pm \frac{J}{2r^2\Delta}dr, \tag{5.16}$$

c.f. equations (4.9)-(4.13), they correspond to:

$$\phi_{\pm} = \text{const.} \tag{5.17}$$

The metric (5.1) in the generalized Eddington-Finkelstein coordinates is:

$$ds^2 = -\Delta dt_{\pm}^2 + \frac{1}{4r^2}[Jdt_{\pm} - 2r^2 d\phi_{\pm}]^2 \pm 2dt_{\pm}dr, \tag{5.18}$$

so that the induced metric on the world-sheets $\Sigma_\pm$ becomes:

$$dS^2 = -F dt_\pm^2 \pm 2dt_\pm dr. \tag{5.19}$$

It must be stressed that the metrics (5.19) for Σ_+ and Σ_- simply describe different portions of global $1+1$ dimensional anti de Sitter spacetime. Contrary to their $2+1$ dimensional origin, the $2+1$ BH-ADS, the principal Killing surfaces $\Sigma_\pm$ are thus purely "cosmological" in nature; the 2-D horizon at $r = \sqrt{M}l$ is merely of Rindler-type.

For $J = 0$, $\Sigma_\pm$ are geodesic surfaces in the 3-D spacetime and they describe the two branches of a geodesically complete 2-D manifold ($1+1$ anti de Sitter spacetime). However, for the generic BH-ADS geometry ($J \neq 0$), only one of two null basic lines of the principal Killing surface, namely the ray $\gamma_\pm$ with tangent vector $l_\pm$, is geodesic in the three-dimensional embedding space. The other basic null ray γ' is geodesic in $\Sigma_\pm$ but not in the embedding space. This implies that in general (when $J \neq 0$) the principal Killing surface is not geodesic. Furthermore, it can be shown that $\Sigma_\pm$ considered as a 2-D manifold is geodesically incomplete with respect to its null geodesic γ'.

As a consequence of $\Sigma_\pm$ not being geodesic (when $J \neq 0$), it is possible to send causal signals from the interior of the 2-D horizon to its exterior (as seen by an observer using the coordinates (5.19)) by exploiting the extra dimension of the 3-D spacetime. It is evident that there exist causal lines leaving the ergosphere and entering the exterior of the BH-ADS spacetime. It means that the "interior" and "exterior" of Σ_+ (as seen by an observer using the coordinates (5.19)) can be connected by 3-D causal lines. It can also be shown that the causal line can be chosen to be a null geodesic. The argument is similar to the situation in the 4-D Kerr-Newman spacetime [13] so we shall not repeat it here.

It is an interesting observation that the horizon of Σ_+ (as seen by an

observer using the coordinates (5.19)) coincides with the static limit of the 3-D rotating black hole. The 2-D surface gravity, which is proportional to the 2-D temperature, is given by:

$$\kappa^{(2)} = \frac{1}{2}\frac{dF}{dr}\bigg|_{r=r_{st}} = \frac{\sqrt{M}}{l}. \tag{5.20}$$

The surface gravity of the 3-D BH-ADS spacetime is [7]:

$$\kappa^{(3)} = \frac{r_+^2 - r_-^2}{l^2 r_+}, \tag{5.21}$$

where $r_\pm$ are defined in equation (5.4). It can then be easily shown that:

$$\kappa^{(2)} \geq \kappa^{(3)}. \tag{5.22}$$

That is to say, the 2-D temperature is higher than the 3-D temperature (except when $J = 0$) and it is always positive. Even if the 3-D black hole is extreme, the 2-D temperature is non-zero. It should be stressed that the temperatures discussed here are the temperatures as measured by an observer situated where the timelike Killing vector is normalized such that $\xi^2 = -1$, [25] that is at $r = l\sqrt{M+1}$.

We close this section with the following remark:

The physical properties of the Principal Killing Surface Σ_+ can be investigated by considering the propagation of perturbations (stringons) along the cosmic string. The general equations determining the perturbations were obtained at the end of Section 4, equations (4.25)-(4.26). For Σ_+, with metric given by (5.19), the equation determining the perturbations is:

$$\left(\Box - \frac{2}{l^2}\right)\Phi = 0. \tag{5.23}$$

It is convenient to make the following coordinate transformations. We first introduce the coordinates $(\tilde{u}, \tilde{r})$:

$$u = \tilde{u} + \tilde{r}, \qquad r = -\sqrt{M}l \,\text{cotanh}\left(\frac{\sqrt{M}\,\tilde{r}}{l}\right), \tag{5.24}$$

and then (λ, ρ) :

$$\tilde{r} = \sqrt{M}l \,\text{sec}\rho \cos\lambda, \qquad \tanh\left(\frac{\sqrt{M}\tilde{u}}{l}\right) = \frac{\sin\rho}{\sin\lambda}, \tag{5.25}$$

so that the line element (5.19) takes the standard $1+1$ anti de Sitter form:

$$dS^2 = l^2 \text{sec}^2\rho \left(-d\lambda^2 + d\rho^2\right), \tag{5.26}$$

and the d'Alambertian is given by:

$$\Box = \frac{\cos^2\rho}{l^2}\left(-\partial_\lambda^2 + \partial_\rho^2\right). \tag{5.27}$$

In this formalism, the classical and quantum aspects of equation (5.23) have been discussed in the literature, so we shall not go into the details here. We refer the interested reader to Ref. [27], and references given therein.

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A Explicit form of $\xi_{\mu;\nu}$

Using the metric (2.1), it is straightforward to compute the Christoffel symbols in terms of (F, A_i, H_{ij}). It can then be shown that the anti-symmetric

tensor $\xi_{\mu;\nu}$ takes the form:

$$\xi_{\mu;\nu} = \begin{pmatrix} 0 & -\frac{1}{2}F_{,i} \\ \frac{1}{2}F_{,i} & (FA_{[j})_{,i]} \end{pmatrix}. \tag{A.1}$$

The eigenvalues λ are obtained from the equation:

$$\det(\xi_{\mu;\nu} - \lambda g_{\mu\nu}) = 0, \tag{A.2}$$

and leads to:

$$\lambda = 0 \quad \text{or} \quad \lambda = \mp\frac{1}{2}[H^{ij}F_{,i}F_{,j} - F^4(e^{ij}A_{i,j})^2]^{1/2} \equiv \mp\kappa. \tag{A.3}$$

Here H^{ij} and e^{ij} are defined by:

$$H_{ik}H^{jk} = \delta_i^j, \qquad e^{ij} = \frac{\epsilon^{ij}}{\sqrt{\det(H)}}, \tag{A.4}$$

where ϵ^{ij} is the Levi-Civita symbol in two dimensions. Explicit expressions for the linearly independent eigenvectors $(l_+^\mu, l_-^\mu, m^\mu)$ can then also be obtained in general (for $\kappa \neq 0$), but they are not important here.

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